PDE QUALIFYING EXAMINATION (MAY 2018)

Do not use any outside resources or collaborate with anyone to help you complete this exam. You have 2 hours to complete this exam.

Problem 1. Fix $n < p \leq \infty$.

(a) Show there is exactly one unique value of $q > 0$ (and find that value), such that there can exist a constant $C > 0$ depending only on n, q, and p such that

$$\sup_{\mathbb{R}^n} |u| \leq C(\mathrm{Vol}(\operatorname{spt} u))^q \|Du\|_{L^p(\mathbb{R}^n)}, \quad \forall\, u \in C_c^\infty(\mathbb{R}^n),$$

(you do not have to prove the inequality itself).

(b) Prove there **cannot** exist a constant $C > 0$ independent of u such that

$$\sup_{\mathbb{R}^n} |u| \leq C\|Du\|_{L^p(\mathbb{R}^n)}, \quad \forall\, u \in C_c^\infty(\mathbb{R}^n).$$

Problem 2. Let $\Omega \subset \mathbb{R}^n$ be bounded and connected with C^1 boundary, $n \geq 2$, and $g \in L^2(\partial\Omega)$, $f \in L^2(\Omega)$ be fixed. We say $u \in H^1(\Omega)$ is a weak solution of the *nonhomogeneous Neumann problem* if for all $v \in H^1(\Omega)$,

$$\sum_{i=1}^n \int_\Omega u_i v_i = \int_{\partial\Omega} g \cdot Tv dS + \int_\Omega fv$$

where $T : H^1(\Omega) \to L^2(\partial\Omega)$ is the trace operator. Prove that there exists a weak solution to the above problem for any g and f satisfying

$$\int_{\partial\Omega} g dS + \int_\Omega f = 0.$$

Problem 3. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open, connected set with C^1 boundary, $n \geq 2$, and fix $1 \leq p < \infty$. Also define $\mathcal{W} := \{u \in W^{1,p}(\Omega) \mid \int_{\partial\Omega} Tu\, dS = 0\}$ where $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is the trace operator. Prove there is a $C > 0$ such that

$$\|u\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)}.$$

(Hint: try to mimic the proof of the Poincaré-Wirtinger inequality).

Problem 4. Let $\Omega \subset \mathbb{R}^n$ be an open domain. Prove that if u, $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ are such that $u \leq w$ on $\partial\Omega$ and

$$-\sum_{i=1}^n (2 - \cos(|Du|^2))u_{ii} + u(|Du|^3 + 1) \leq -\sum_{i=1}^n (2 - \cos(|Dw|^2))w_{ii} + w(|Dw|^3 + 1)$$

on Ω, then $u \leq w$ on all of Ω.

Problem 5. Let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < 1 - x^2\}$, and define the divergence form operator

$$Lu := -(a^{ij} u_i)_j,$$

$$(a^{ij}) = \begin{pmatrix} 3 & x^2 + y \\ x^2 + y & 3 \end{pmatrix}$$

Prove that if λ_1 is the principal eigenvalue for the Dirichlet problem associated to L on Ω, in the **weak** sense, then $\lambda_1 \geq \frac{1}{2}$.